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ACOUSTICAL BIREFRINGENCE OF ULTRASONIC WAVES IN DEFORMED ISOTROPIC ELASTIC MATERIALS

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Abstract—The basic relations of the acoustoelasticity are deduced by means of the infinitesimal wave propagation in a deformed isotropic elastic material. The propagation equation is characterised by the acoustical tensor. This results in the three polarization directions, which are perpendicular to each other and, in general, do not coincide with the principal axes of the stress. When the propagation direction coincides with one of the principal axes of the stress, the former are identical to the latter and the so-called stress-acoustical law holds, which denotes that the magnitude of the acoustical birefringence is proportional to the difference of the secondary principal stresses.

1. INTRODUCTION

As an experimental nondestructive stress analysis, the polarized transverse ultrasonic waves have provided a new method of measurements, which is christened the acoustoelasticity. Benson and Raelson [1] proposed this method and reported the experimental data of the stress in simple compression and the residual stress.

The second and third order elastic coefficients influence the propagation velocity of the wave in a hydrostatically deformed materials and these are measured by means of the ultrasonics [2–4].

Hayes and Rivlin [5] investigated the surface waves in an isotropic elastic material in finite deformations. Toupin and Bernstein [6] obtained the relations of the acoustoelastic effect and determined the third-order elastic constants of an isotropic material. Thurston [7, 8] discussed, in general, the wave propagation in a strained material. But all of the above investigations are restricted to the case of homogeneous strain state.

In this paper, by the same method of Pearson [9], the stress is calculated in the state which is superposed by an infinitesimal displacement on an initially deformed state, and the propagation equation is deduced by the order estimation. Then the stress-acoustical law is determined.

2. BASIC RELATIONS OF THE THEORY OF INFINITESIMAL ELASTIC DEFORMATIONS SUPERIMPOSED ON A SMALL ELASTIC DEFORMATION

The elastic material of the Green type is deformed from the natural state I to the static state II. We consider that the state III of propagation of waves differs infinitesimally from the known state II. The positions with respect to a rectangular Cartesian coordinates are specified by X_k, x_k and x'_k $(k = 1, 2, 3)$ in I, II and III respectively and

$$x_k = X_k + u_k, \qquad x'_k = x_k + w_k, \tag{2.1}$$

where u_k is the static displacement vector and the time dependent displacement vector w_k is assumed to be infinitesimal, and thus in the following calculations the second and the higher order terms of w_k will be neglected.

The stress tensor in the state II of the hyperelastic compressible material is exactly denoted by

$$t_{kl} = \frac{\rho}{\rho_0}\frac{\partial \Sigma}{\partial E_{mn}}\frac{\partial x_k}{\partial X_m}\frac{\partial x_l}{\partial X_n}, \tag{2.2}$$

where ρ_0 and ρ are the density in I and II respectively, Σ is the energy function of the given material and

$$E_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_l}+\frac{\partial u_l}{\partial X_k}+\frac{\partial u_m}{\partial X_k}\frac{\partial u_m}{\partial X_l}\right) \tag{2.3}$$

is the Lagrangian strain tensor in II. Here and in the following the summation convention is applied to every repeated suffix.

The deviation of the strain tensor δE_{kl} from the state II to III is expressed as

$$\delta E_{kl} = E'_{kl}-E_{kl} = \frac{1}{2}\left(\frac{\partial x_m}{\partial X_k}\frac{\partial w_m}{\partial X_l}+\frac{\partial x_m}{\partial X_l}\frac{\partial w_m}{\partial X_k}\right), \tag{2.4}$$

where E'_{kl} is the strain tensor in III and we indicate any quantities in the state III by marking the prime. By means of the simple calculations, we have the stress tensor in the state III:

$$\begin{aligned} t'_{kl} &= \frac{\rho'}{\rho_0}\left(\frac{\partial \Sigma}{\partial E_{mn}}\right)'\frac{\partial x'_k}{\partial X_m}\frac{\partial x'_l}{\partial X_n} \\ &= \frac{\rho'}{\rho_0}\left(\frac{\partial \Sigma}{\partial E_{mn}}+\frac{\partial^2 \Sigma}{\partial E_{mn}\partial E_{pq}}\delta E_{pq}\right)\left(\frac{\partial x_k}{\partial X_m}+\frac{\partial w_k}{\partial X_m}\right)\left(\frac{\partial x_l}{\partial X_n}+\frac{\partial w_l}{\partial X_n}\right) \\ &= \left(1-\frac{\partial w_m}{\partial x_m}\right)t_{kl}+t_{km}\frac{\partial w_l}{\partial x_m}+t_{ml}\frac{\partial w_k}{\partial x_m}+S_{klrs}\frac{\partial w_r}{\partial x_s}, \end{aligned} \tag{2.5}$$

where

$$S_{klrs} = \frac{\rho}{\rho_0}\frac{\partial^2 \Sigma}{\partial E_{mn}\partial E_{pq}}\frac{\partial x_k}{\partial X_m}\frac{\partial x_l}{\partial X_n}\frac{\partial x_r}{\partial X_p}\frac{\partial x_s}{\partial X_q}. \tag{2.6}$$

The equilibrium equation in the static state II and the equation of motion in III are given respectively

$$\frac{\partial t_{kl}}{\partial x_l} = 0 \qquad \text{and} \qquad \frac{\partial t'_{kl}}{\partial x'_l} = \rho' \ddot{w}_k, \tag{2.7}$$

where no body force is assumed. Substituting (2.5) into $(2.7)_2$ and referring $(2.7)_1$ and

$$\frac{\partial}{\partial x'_l} = \frac{\partial}{\partial x_l}-\frac{\partial w_m}{\partial x_l}\frac{\partial}{\partial x_m},$$

we have

$$t_{lm}\frac{\partial^2 w_k}{\partial x_l \partial x_m}+\frac{\partial}{\partial x_l}\left[S_{klrs}\frac{\partial w_r}{\partial x_s}\right] = \rho \ddot{w}_k. \tag{2.8}$$

Now we assume that the deformation and the rotation of a body element are small and therefore the second and the higher order terms of the displacement gradients may be neglected with respect to the first order [10]. Then the Lagrangian strain tensor is reduced to the usual one

$$e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial X_l} + \frac{\partial u_l}{\partial X_k}\right). \tag{2.9}$$

The isotropic linear elastic material has the energy function

$$\Sigma = \tfrac{1}{2}(\lambda + 2\mu)\mathrm{I}_e^2 - 2\mu \mathrm{II}_e, \tag{2.10}$$

where λ and μ are the Lamé constants, and $\mathrm{I}_e \equiv e \equiv e_{kk}$ and $\mathrm{II}_e \equiv \frac{1}{2}(\mathrm{I}_e^2 - e_{kl}e_{lk})$ are the first and second strain invariants respectively with respect to e_{kl}. The energy function (2.10) assures the Hooke's law:

$$t_{kl} = 2\mu e_{kl} + \lambda e \delta_{kl},$$

or

$$e_{kl} = \frac{1}{2\mu}\left(t_{kl} - \frac{\lambda}{3\lambda + 2\mu} t\delta_{kl}\right), \tag{2.11}$$

$$e = \frac{1}{3\lambda + 2\mu} t, \qquad t \equiv t_{kk}.$$

Within the first order of the displacement gradient, we can easily calculate that

$$\begin{aligned} S_{klrs}\frac{\partial w_r}{\partial x_s} = {} & (1-e)\left[\mu\left(\frac{\partial w_k}{\partial x_l} + \frac{\partial w_l}{\partial x_k}\right) + \lambda \frac{\partial w_m}{\partial x_m}\delta_{kl}\right] \\ & + 2\mu\left[e_{km}\left(\frac{\partial w_m}{\partial x_l} + \frac{\partial w_l}{\partial x_m}\right) + e_{ml}\left(\frac{\partial w_k}{\partial x_m} + \frac{\partial w_m}{\partial x_k}\right)\right] \\ & + 2\lambda\left(e_{kl}\frac{\partial w_m}{\partial x_m} + e_{mn}\frac{\partial w_m}{\partial x_n}\delta_{kl}\right). \end{aligned} \tag{2.12}$$

A propagating sound wave in a nonhomogeneously deformed material is not a plane wave, which has the same direction of propagation and the same amplitude in all space. But where the strain gradient is not so large, the sound wave is regarded as plane in any small region of space and the amplitude and the direction of propagation should vary only slightly over distances of the order of the wavelength.

If this condition holds, the terminology of geometrical acoustics may be used. We write the infinitesimal displacement as

$$w_k = \overline{W}_k e^{i\psi} \tag{2.13}$$

where the amplitude $\overline{W}_k$ is a slowly varying function of the co-ordinates and time, while the wave phase ψ, which is called the eikonal, may be expanded in series

$$\psi = \psi_0 + \mathbf{k}\,.\,\mathbf{x} - \omega t \tag{2.14}$$

up to terms of the first order, where we define the wave vector and the frequency at each point as

$$\mathbf{k} = \mathbf{grad}\,\psi, \qquad \omega = -\frac{\partial \psi}{\partial t}. \tag{2.15}$$

The gradient of w_k is then approximated by the multiplication by **k**. If the ultrasonic frequency is larger than 5×10^6 c/s and the propagation velocities of the transverse waves in iron or in aluminium are about 3×10^5 cm/sec, then the magnitude of the wave vector $k = \omega/v$ is larger than 10^2/cm and the wave length is less than 0·6 mm. On the other hand, for a distance of one wave-length, the strain may be assumed not to change its magnitude tolerably in the usual deformation state. Therefore we may neglect the terms of the derivatives of the strain in the left-hand side of (2.8).

Without loss of generality we may adopt the direction of the wave vector as x_3-axis. Expressing the strains by the stress according to $(2.11)_2$ and $(2.11)_3$, the equation of motion is then reduced to

$$A_{kl}w_l = \rho\left(\frac{\omega}{k}\right)^2 w_k = \rho v^2 w_k, \tag{2.16}$$

where

$$v = \frac{\omega}{k} \tag{2.17}$$

is the propagation velocity and

$$\begin{aligned} A_{kl} \equiv & \left(\mu + 2t_{33} - \frac{2\lambda+\mu}{3\lambda+2\mu}\, t\right)\delta_{kl} + t_{kl} \\ & + \frac{\lambda+\mu}{\mu}(t_{k3}\delta_{l3} + \delta_{k3}t_{l3}) \\ & + \frac{\lambda+\mu}{\mu}\left(\mu - \frac{2\lambda+\mu}{3\lambda+2\mu}\, t\right)\delta_{k3}\delta_{l3} \end{aligned} \tag{2.18}$$

is called the acoustical tensor.

3. POLARIZATION DIRECTIONS, PROPAGATION VELOCITIES AND ACOUSTICAL BIREFRINGENCE

The symmetric tensor $\|A_{kl}\|$ has in general three mutually perpendicular principal directions [11] and then the three polarization directions of the sound wave exist and they are perpendicular each other. The propagation velocities of the polarized wave correspond to the eigenvalues of the matrix $\|A_{kl}\|$.

We assume that the direction of the wave vector coincides with a principal axis of the stress tensor, i.e. $t_{13} = t_{23} = 0$ and $t_{33} = t_3$.

In this case, we have

$$\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 + \mathbf{A}_2, \tag{3.1}$$

where

$$\mathbf{A_0} \equiv \mu \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\lambda+2\mu}{\mu} \end{bmatrix},$$

$$\mathbf{A_1} \equiv \left(2t_3 - \frac{2\lambda+\mu}{3\lambda+2\mu} t\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{\lambda+2\mu}{\mu} \end{bmatrix}, \tag{3.2}$$

$$\mathbf{A_2} \equiv \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_3 \end{bmatrix}.$$

When the stress in a material is absent, the acoustical tensor is reduced to $\mathbf{A_0}$ and the wave velocities are

$$v_{0\parallel} = \left(\frac{\lambda+2\mu}{\rho_0}\right)^{\frac{1}{2}} \quad \text{and} \quad v_{0\perp} = \left(\frac{\mu}{\rho_0}\right)^{\frac{1}{2}}. \tag{3.3}$$

The former corresponds to one longitudinal dilatational wave and the latter to two transverse shear waves whose directions are arbitrary in the wave front Π.

When a material is stressed, the velocity of the longitudinal wave takes the value

$$v_{\parallel} = \left\{ \frac{\lambda+2\mu}{\rho}\left[1 + \frac{1}{\mu}\left(2t_3 - \frac{2\lambda+\mu}{3\lambda+2\mu} t\right) + \frac{t_3}{\lambda+\mu}\right]\right\}^{\frac{1}{2}}. \tag{3.4}$$

Transforming the coordinates into the principal axes of the stress in the plane Π, the matrix $\mathbf{A_2}$ takes the form

$$\begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}, \tag{3.5}$$

where t_1 and t_2 are the principal values of the stress in Π. Thus the shear waves are polarized into the principal axes of the stress and their velocities are

$$v_\alpha = \left\{ \frac{\mu}{\rho}\left[1 + \frac{1}{\mu}\left(2t_3 - \frac{2\lambda+\mu}{3\lambda+2\mu} t + t_\alpha\right)\right]\right\}^{\frac{1}{2}}, \quad (\alpha = 1, 2). \tag{3.6}$$

When an arbitrarily polarized shear ultrasonic wave is inserted into a deformed metal specimen in a two-dimensional stress state, the wave is separated into two linearly polarized waves, whose directions are the principal axes of the stress.

Their propagation velocities are given by (3.6) and they are functions of the stress state.

When the wave reaches to the other side of the specimen, the phase difference of two components will occur. According to the same concept of the photoelasticity, the stress state is indicated by the fringe-order per unit length (see, e.g., Frochot [12])

$$N = \frac{\omega}{2\pi}\left(\frac{1}{v_2} - \frac{1}{v_1}\right). \tag{3.7}$$

The second term in the square bracket of (3.6) is less than 10^{-3} under the proportional limit, then we have in good approximation that

$$N = \alpha(t_1 - t_2), \tag{3.8}$$

where

$$\alpha = \frac{\omega}{4\pi\rho v_{0\perp}^3} \tag{3.9}$$

may be called as the acoustoelastic sensitivity. The relation (3.8) indicates that *stress-acoustical law* holds, i.e. the magnitude of the acoustical birefringence is proportional to the difference of the secondary principal stresses.

4. DISCUSSIONS

The acoustical birefringent phenomena in deformed isotropic elastic materials are formulated by a simple analytical method.

With respect to the acceleration waves, Truesdell [13] and Truesdell and Noll [14] investigated the most general theories and the properties of the propagation waves are completely formulated. But with respect to the sinusoidal plane progressive wave, all of the reported papers [5–8, 13, 14] are restricted in a material subject to homogeneous strain. Here we treated the infinitesimal sinusoidal wave propagation superimposed on a small nonhomogeneous deformation and obtained the propagating equation by means of the acoustical eikonal and the numerical order estimation.

The stress-acoustical law are determined, which denotes that the acoustical birefringence is proportional to the difference of the transversal principal stresses. The law may be useful as the experimental stress analysis using the ultrasonic waves.

REFERENCES

[1] R. W. BENSON and V. J. RAELSON, *Prod. Eng* **30**, 56 (1956).
[2] T. BATEMAN, W. P. MASON and H. J. MCSKIMIN, *J. appl. Phys.* **32**, 928 (1961).
[3] D. S. HUGHES and J. L. KELLY, *Phys. Rev.* **92**, 1145 (1953).
[4] W. B. DANIELS and CHARLES S. SMITH, *Phys. Rev.* **111**, 713 (1958).
[5] M. HAYES and R. S. RIVLIN, *Archs ration. Mech. Analysis* **8**, 358 (1961).
[6] R. A. TOUPIN and B. BERNSTEIN, *J. acoust. Soc. Am.* **33**, 216 (1961).
[7] R. N. THURSTON, *Physical Acoustics*, edited by WARREN P. MASON, Vol. 1, Part A, Chapter 1. Academic Press (1964).
[8] R. N. THURSTON, *Proc. IEEE* **10**, 1320 (1965).
[9] CARL E. PEARSON, *Q. appl. Math.* **14**, 133 (1956).
[10] V. V. NOVOZHILOV, *Foundations of the Nonlinear theory of Elasticity*. Graylock Press (1953).
[11] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Vol. 1, Chapter I. Interscience (1953).
[12] M. M. FROCHT, *Photoelasticity*, Vol. 1, Chapter III. Wiley (1962).
[13] C. TRUESDELL, *Archs ration. Mech. Analysis* **8**, 263 (1961).
[14] C. TRUESDELL and W. NOLL, *The Non-Linear Field Theories of Mechanics, Handbuch der Physik*, edited by S. FLÜGGE, Vol. **III/3**, D, I, e. Springer (1965).

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Абстракт Выводятся основные зависимости акустики улругой сплошной среды в смысле инфинитезималъного распространения волны в деформируемым изотропном упругом материале. Уравнение распространения характеризуется акустическим тензором. Зти резулътаты в трех налравлениях поляризации, взаимно перпендикулярным, вообще не совпадают с главными осями напряжений. Когда форма направления распрос транения согласовывается с одним из главных направлений напряжения, тогда предыдущие резулвтаты являютсн одинаковыми с последующими и обязывает так называемый закон акустического напряжения. Это обозначает, что величина акустического двойного преломления пропорционалвна разнице второстепенных главных напряжений.